RANDOM WALK ON KNOT DIAGRAMS, COLORED JONES POLYNOMIAL AND IHARA-SELBERG ZETA FUNCTION

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ABSTRACT. A model of random walk on knot diagrams is used to study the Alexander polynomial and the colored Jones polynomial of knots. In this context, the inverse of the Alexander polynomial of a knot plays the role of an Ihara-Selberg zeta function of a directed weighted graph, counting with weights cycles of random walk on a 1-string link whose closure is the knot in question. The colored Jones polynomial then counts with weights families of "self-avoiding" cycles of random walk on the cabling of the 1-string link. As a consequence of such interpretations of the Alexander and colored Jones polynomials, the computation of the limit of the renormalized colored Jones polynomial when the coloring (or cabling) parameter tends to infinity whereas the weight parameter tends to 1 leads immediately to a new proof of the Melvin-Morton conjecture, which was first established by Rozansky and by Bar-Natan and Garoufalidis.

1. INTRODUCTION

The Alexander polynomial and the Jones polynomial, both characterized by simple crossing change formulae, are probably the two most celebrated invariants in knot theory. While the Alexander polynomial appears again and again in different contexts, making us feel quite comfortable with it, the nature of the Jones polynomial remains mysterious. In this paper, we will provide a new perspective for the study of the Jones polynomial (and its generalizations – the so-called colored Jones polynomial), the Alexander polynomial and their relationship. An immediate outcome of this new perspective is a straightforward proof of the Melvin-Morton conjecture.

In [7], a model of random walk on knot diagrams was introduced. When we were seeking formulations of the Alexander and Jones polynomials in this model of random walk, a paper of Foata and Zeilberger [4] caught our attention. In that paper, Foata and Zeilberger established a general combinatorial framework for counting with weights Lyndon words in a free monoid generated by a totally ordered set, one of its consequences is a proof of Bass' evaluations of the Ihara-Selberg zeta function for graphs. We noticed that one of the main theorems of [4] implies the following fact:

Take a 1-string link and consider all families of cycles on this 1-string link in our model of random walk. Every cycle is assigned with a weight (probability). Then the Ihara-Selberg type zeta function constructed using these weights is equal to the inverse of the Alexander polynomial of the knot obtained as the closure of the 1-string link, up to a factor in the form of a power of the weight parameter.

There is a remarkable relation between the colored Jones polynomial and the Alexander polynomial, which was first noticed and conjectured by Melvin and Morton [8]. Rozansky [10] gave an argument for this conjecture using the Chern-Simons path integral formalism of the colored Jones polynomial and the relation between Ray-Singer analytic torsion and the Alexander polynomial. The rigorous proof of the Melvin-Morton conjecture was given by Bar-Natan and Garoufalidis [2], using the full power of the theory of finite type knot invariants.

In our setting of random walk on knot diagrams, the Jones polynomial counts only simple families of cycles on the 1-string link, i.e. families of cycles which do not share any edge. To take into account of all cycles, we have to use the colored Jones polynomial. A state sum formula for the (renormalized) colored Jones polynomial with the coloring parameter $d + 1$ implies that it counts simple families of cycles on d-cabling of the 1-string link in question. To relate the colored Jones polynomial with the Alexander polynomial, we lift families of cycles on the string link to its d-cabling with the weight parameter adjusted appropriately. A family of cycles on the 1-string link can have many liftings to its cabling. Weights of all liftings add up to the weight of the original family of cycles, whereas the weights of non-simple liftings vanish in the limit when $d \to \infty$. So in the limit, only weights of simple families of cycles survive and this calculation leads to a proof of the Melvin-Morton conjecture.

We remark that our formulation of the limit of the colored Jones polynomial is analogous to the limit of partition functions on a finite lattice with a fixed boundary condition in statistical mechanics. Our proof of the Melvin-Morton conjecture is in spirit close to Rozansky's proof using the semi-classical limit of Chern-Simons path integral.

The model of random walk on knot diagrams has a much richer content than we have touched upon here. A more detailed exploration of this model will be the subject of our future publications.

2. Random walk on knot diagrams

2.1. **Wirtinger presentation and free derivatives.** Fix an oriented knot diagram K, we will label the arcs in the knot diagram separated by crossings at the under-crossed strands using the letters $x_1, x_2, \ldots, x_n$. The knot group $G(K) = \pi_1(\mathbb{R}^3 \setminus K)$ admits a Wirtinger presentation as follows: It has $x_1, x_2, \ldots, x_n$ as generators, and

one relation for each crossing. If a crossing has incident arcs x_i, x_j, x_k, where x_i separates x_j and x_k in a small neighborhood of the crossing and the knot orientation points x_j toward x_k, the relation is

$$x_j = x_i^\epsilon x_k x_i^{-\epsilon}.$$

Here $\epsilon = \pm 1$ is the sign of the crossing.

With respect to the abelianization $\phi : \mathbb{Z}G(K) \to \mathbb{Z}[t^{\pm 1}]$, sending each x_i to t, a free derivative $\partial : \mathbb{Z}G(K) \to \mathbb{Z}[t^{\pm 1}]$ is a linear map such that

$$\partial(g_1 g_2) = \partial(g_1) + \phi(g_1)\partial(g_2) \qquad \text{for all } g_1, g_2 \in G(K).$$

The $\mathbb{Z}[t^{\pm 1}]$-module of free derivatives on the free group F generated by $x_1, x_2, \ldots, x_n$ is spanned by ∂_i, $i = 1, 2, \ldots, n$ with $\partial_i(x_j) = \delta_{ij}$. Let ∂ be a free derivative on $G(K)$. Then $\partial = \sum_{i=1}^n A_i \partial_i$ as a free derivative on F, where $A_i \in \mathbb{Z}[t^{\pm 1}]$, and it has to satisfy the relation

$$\partial(x_j) = t^\epsilon \partial(x_k) + (1 - t^\epsilon)\partial(x_i)$$

for each Wirtinger relation $x_j = x_i^\epsilon x_k x_i^{-\epsilon}$. Thus the $\mathbb{Z}[t^{\pm 1}]$-module of free derivatives on $G(K)$ can be thought of as generated by the symbols A_i, $i = 1, 2 \ldots, n$ and subject to the relation

$$A_j = t^\epsilon A_k + (1 - t^\epsilon)A_i$$

for each Wirtinger relation $x_j = x_i^\epsilon x_k x_i^{-\epsilon}$.

We define an $n \times n$ matrix $\tilde{\mathcal{B}}$ as follows. The j-th row of $\tilde{\mathcal{B}}$ has at most two non-zero entries: for each relation $A_j = t^\epsilon A_k + (1 - t^\epsilon)A_i$, when $k \neq i$, the (j, k)-entry is t^ϵ and the (j, i)-entry is $1 - t^\epsilon$; when $k = i$, the only non-zero entry is the (j, k)-entry, which is equal to 1.

Let $\mathcal{B}$ be the $(n-1) \times (n-1)$ matrix obtained from $\tilde{\mathcal{B}}$ by deleting the first row and the first column. Then $\det(I - \mathcal{B})$ is the Alexander polynomial of the knot K (recall that the Alexander polynomial of a knot is only defined up to powers of t). In fact, this is always true no matter which j-th row and column are deleted.

2.2. A model of random walk on knot diagrams.

In our model of random walk on the knot diagram K, we take $\{A_1, A_2, \ldots, A_n\}$ to be the space of states. The transition matrix is simply $\tilde{\mathcal{B}}$. This is obviously a stochastic matrix in the sense that the entries in each row add up to 1. In the case when all crossings of K are positive (K is a positive knot diagram), we get a genuine Markov chain for each $t \in [0, 1]$. Otherwise, we may have negative probabilities for negative crossings.

In this model of random walks on K, a path from A_i to A_j is a sequence of transitions of states from A_i to A_j. Each such path is associated with a weight ("probability"), which is the product of "transition probabilities" along this path. Pick a state, say A_1, consider paths from A_1 to itself which will not contain A_1 at any intermediate

stage, i.e. we consider paths of first return from A_1. Equivalently, we may regard A_1 as being broken into two states A_1' and A_1'', one initial and one terminal. This can be done by breaking the arc x_1 into two arcs x_1' and x_1'' and changing the knot K into a 1-string link T. Then we consider all paths on T from A_1' (the bottom of T) to A_1'' (the top of T).

Proposition 2.1. *The summation of weights over all paths on T from A_1' to A_1'' is equal to 1.*

Proof. To calculate the sum of weights of all paths from A_1' to A_1'' amounts to solve the system of linear equations

$$A_j = t^\epsilon A_k + (1 - t^\epsilon) A_i$$

for A_1'' with $A_1' = 1$ given. We have the unique solution $A_1'' = 1$. For more details of the proof, see [7].   □

We have the following theorem.

Theorem 2.2. *1. Let K be a positive knot diagram with n arcs. Then for every pair (i, j), there is an integer $m \leq n$, such that the (i, j)-entry of the matrix $\tilde{\mathcal{B}}^m$ is positive. Hence, the Markov chain is irreducible.*

2. Let $p_{i,j}^{(k)}$ be the (i, j)-th entry of $\tilde{\mathcal{B}}^k$. For each $t \in [0, 1]$ and i, j, $\sum_{k=1}^{\infty} p_{i,j}^{(k)} = \infty$. Hence each state is persistent.

Proof. 1. This is true because we can travel along the knot from any state A_i to A_j in $\leq n$ steps.

2. If $i = j$, by Proposition 2.1, if we sum the weights of all the k-th return paths for $1 \leq k \leq n$, the sum is n. For $i \neq j$, the sum $\sum_{k=1}^{n} p_{i,j}^{(k)} > n$.   □

Imagine that a ball travels on the knot diagram in the direction specified by the orientation of the knot. It will make a choice when it comes to an ϵ-crossing from the under-crossed segment: it may either jump up with probability $1 - t^\epsilon$ and keep traveling on the over-crossed segment or keep traveling with probability t^ϵ on the under-crossed segment. This is an intuitive picture of our model of random walk on knot diagrams. We will call this model the "jump-up" model. There is also a "dual" model of jump-down random walk on knot diagrams. In this model, one needs to make a choice at the over-crossed segment of a crossing: jump-down or keep traveling. There are some delicate connections and differences between these two models which we will not discuss here. We only notice that the two random walk models correspond to different choices of base points in the Wirtinger presentation.

2.3. **State sum for the Jones polynomial.** State sum models on knot diagrams is one of the main tools attained in the development of topological quantum field theories. The state model we will use for the Jones polynomial is given by Turaev in [11] based on earlier constructions of Jones. For this model, we need an R-matrix. The R-matrix of $\mathfrak{sl}(2)$ with respect to the fundamental representation is given as follows (with $\bar{q} = q^{-1}$ and $\bar{R} = R^{-1}$):

$$R_{0,0}^{0,0} = R_{1,1}^{1,1} = -q, \ R_{0,1}^{1,0} = R_{1,0}^{0,1} = 1, \ R_{0,1}^{0,1} = \bar{q} - q,$$
$$\bar{R}_{0,0}^{0,0} = \bar{R}_{1,1}^{1,1} = -\bar{q}, \ \bar{R}_{0,1}^{1,0} = \bar{R}_{1,0}^{0,1} = 1, \ \bar{R}_{1,0}^{1,0} = q - \bar{q},$$

and all other entries of the R-matrix are zero.

In this model, we consider the 1-string link T as a planar graph by looking at its projection. A state s is an assignment of 0 or 1 to each edge of the graph. For each vertex (crossing) v, if a, b, c, d are edges incident to v, define

$$\pi_v(s) = (R^\epsilon)_{s(a),s(b)}^{s(c),s(d)},$$

where ϵ is the sign of the crossing v, a, b are incoming edges and c, d are outgoing edges. A state s is admissible if $\pi_v(s) \neq 0$ for all vertices v, and the initial and terminal edges having the same assignments. The set of all admissible states will be denoted by $\mathrm{adm}(T)$. We have

$$\mathrm{adm}(T) = \mathrm{adm}_0(T) \amalg \mathrm{adm}_1(T)$$

where $\mathrm{adm}_i(T)$, $i = 0, 1$, is the set of admissible states with $s = i$ on the initial and terminal edges of T. For each admissible state s, define

$$\Pi(s) = \prod_{v:\text{ vertices}} \pi_v(s).$$

Given a 1-string link diagram T, and let K be a closure of T to a knot diagram without introducing any additional crossings, and a state $s \in \mathrm{adm}_i(T), i = 0, 1$. The state s on T can naturally be extended as a state on the knot diagram K. There are quite a few quantities associated with T or the pair (T, s). We will define them here, and these notations will be in force throughout this paper. Also, we will use dashed lines for edges having the assignment 0 in the state s and solid lines for edges having assignment 1.

First we define a modification of diagrams according to a state. *A smoothing of* (T, s) *or* (K, s) is the modification of the diagram by smoothing the crossings marked as

$$\times, \ \times, \ \times, \ \times,$$

we get a collection of circles and an arc in the case of (T, s), and only circles in the case of (K, s). Each circle or arc is marked by 0 or 1.

1. *The writhe of T:* Denote by $\omega(T)$ the writhe, i.e. the summation of signs over all crossings of T.

2. $\beta_i(s), i = 0, 1$: Denote by $\beta_i(s)$ be the sum of signs of crossings whose incident edges all marked by i in s.

3. *Rotation numbers*, $\mathrm{rot}(T), \mathrm{rot}_i(K, s), \mathrm{rot}_i(T, s)$: Smoothing all crossings of T, we get a collection of oriented circles in the plane (together with an oriented arc), and $\mathrm{rot}(T)$ is defined to be the sum of rotation numbers (Whitney's indices) of these circles; For the smoothing of (T, s), the circles are divided into two collections marked by 0 or 1 respectively, and $\mathrm{rot}_i(T, s)$ is defined to be the sum of rotation numbers of the circles marked by i; The definition of $\mathrm{rot}_i(K, s)$ is similar to that of $\mathrm{rot}_i(T, s)$, only that the smoothing of (K, s) has one more circle then (T, s).

For the Jones polynomial $J(K)$, Turaev's state model gives the following formula:

$$J(K) = (-q^2)^{-\omega(T)} \sum_{s \in \mathrm{adm}(T)} q^{\mathrm{rot}_0(K,s) - \mathrm{rot}_1(K,s)} \Pi(s).$$

This formula for the Jones polynomial has the value $q + \bar{q}$ on the unknot, and the standard variable of the Jones polynomial is $t = \bar{q}^2$. It is determined by the following crossing change formula:

$$\bar{t}\, J(K_+) - t\, J(K_-) = (\bar{t}^{\frac{1}{2}} - t^{\frac{1}{2}})\, J(K_0).$$

Remark: This formula is derived from Theorem 5.4 in [11]. The only nontrivial fact is our computation of $\int_D f$ in the formula which is $q^{\mathrm{rot}_0(K,s) - \mathrm{rot}_1(K,s)}$ in our notations. To be more specific, our colors 0,1 correspond to the colors 1,2 in [11], respectively. Also our conventions for rotation numbers are different. Our convention is that the clockwise oriented circle has $\mathrm{rot} = -1$, while the counterclockwise one has $\mathrm{rot} = 1$.

Now let us interpret the state sum from the point view of random walks on knot diagrams. First we take a look at the following table:

$\mathfrak{sl}(2)$	⤬	⤬	⤬	⤬	⤬	⤬	model
⤬	$-q$	$\bar{q} - q$	1	1	0	$-q$	
$-\bar{q}$⤬	1	$1 - \bar{q}^2$	$\bar{q}^2 \cdot (-q)$	$(-\bar{q})$	0	$\bar{q}^2 q^2$	up
$-\bar{q}$⤬	$\bar{q}^2 q^2$	$1 - \bar{q}^2$	$(-\bar{q})$	$\bar{q}^2 \cdot (-q)$	0	1	down
⤬	$-\bar{q}$	0	1	1	$q - \bar{q}$	$-\bar{q}$	
$-q$⤬	1	0	$(-q)$	$q^2 \cdot (-\bar{q})$	$1 - q^2$	$q^2 \bar{q}^2$	up
$-q$⤬	$q^2 \bar{q}^2$	0	$q^2 \cdot (-\bar{q})$	$(-q)$	$1 - q^2$	1	down

Here, as before, a dashed edge has the assignment 0 and a solid edge has the assignment 1. The entry at the row ⤬ (or x⤬) and the column ⤬ is $R_{0,1}^{0,1}$ (or $x R_{0,1}^{0,1}$), etc. The last column indicates two random walk models for this state sum. The two

rows marked by "up" in the last column compare entries of the xR with the weights of the jump-up model, and the two rows marked by "down" compare entries of xR with weights of the jump-down model.

Given a state $s \in \mathrm{adm}_0(T)$, think of the edges with assignments 1 as a collection of cycles that several balls traveled in the jump-up model. Note that their paths may cross transversely but will not pass through the same edge twice. Conversely, if we simultaneously have a few balls traveling on T avoiding the two open arcs, they do not travel over the same edge but may cross transversely, we get a state $s \in \mathrm{adm}_0(T)$ by assigning 1 to all the traveled edges, and 0 otherwise. With such a one-one correspondence, for a state $s \in \mathrm{adm}_0(T)$, we denote by $W_1^\circ(s)$ the product of weights of the collection of cycles formed by edges marked by 1 as cycles in the jump-up model with $t = \bar{q}^2$.

The case of jump-down model is similar, and it corresponds to states in $\mathrm{adm}_1(T)$. Given such a state s, the collection of cycles formed by edges marked by 0 are thought of as cycles in the jump-down model of random walks and $W_0^\circ(s)$ denotes the product of weights.

Lemma 2.3. *In the $\mathfrak{sl}(2)$ state model, we have*

$$\Pi(s) = (-q)^{\omega(T)} q^{2\beta_1(s)} W_1^\circ(s) \qquad for\ s \in \mathrm{adm}_0(T),$$
$$\Pi(s) = (-q)^{\omega(T)} q^{2\beta_0(s)} W_0^\circ(s) \qquad for\ s \in \mathrm{adm}_1(T).$$

Proof. We will show the case $i = 0$. The other case is completely similar. The factor $(-q)^{\omega(T)}$ comes in since we multiply each R-matrix entry at an ϵ-crossing by $(-q^{-\epsilon})$. The term $q^{2\beta_1(s)}$ comes in since we get an extra factor $q^{2\epsilon}$ at a solid ϵ-crossing in the jump-up model. Now using the rows marked by "up" in the table above, we need to show the extra multiplicative factors of $-q^{\pm1}$ inside () in the columns $\times$ and $\times$ will cancel out in the product $\Pi(s)$. Notice that after the modification of T as we did before, the edges marked by 0 is decomposed into a collection of cycles and an arc, having transverse intersections with the cycles formed by edges marked by 1. The intersections between a cycle marked by 1 and a cycle or the path marked by 0 can be paired up. Consider two cases according to whether such a pair makes a contribution to the linking number. In both cases, we see that the extra multiplicative factors of $-q^{\pm1}$ cancel out. $\square$

Denote

$$\int_0^0 (T) = (-q^2)^{-\omega(T)} \sum_{s \in \mathrm{adm}_0(T)} q^{\mathrm{rot}_0(T,s) - \mathrm{rot}_1(T,s)} \Pi(s),$$

$$\int_1^1 (T) = (-q^2)^{-\omega(T)} \sum_{s \in \mathrm{adm}_1(T)} q^{\mathrm{rot}_0(T,s) - \mathrm{rot}_1(T,s)} \Pi(s).$$

Lemma 2.4. *We have $\int_0^0 (T) = \int_1^1 (T)$ and $J(K) = (q + \bar{q}) \int_0^0 (T) = (q + \bar{q}) \int_1^1 (T)$.*

Proof. There are two ways to close up T, both giving the same knot K. Thus, we have

$$J(K) = q \int_0^0 (T) + \bar{q} \int_1^1 (T) = \bar{q} \int_0^0 (T) + q \int_1^1 (T)$$

and this implies the conclusions of the lemma. $\square$

2.4. **Toward a relationship between Jones polynomial and zeta functions.**
Various kinds of zeta functions are basically all about counting of cycles. We may
also express the Jones polynomial in terms of counting "simple families of cycles"
with weights in our model of random walk on a 1-string link T.

Combining previous lemmas, we get the following formula for the Jones polynomial.

Lemma 2.5. *Let K be the closure of a 1-string link T,*

$$J(K) = (q + \bar{q})\, q^{-\omega(T)+\mathrm{rot}(T)} \sum_{s \in \mathrm{adm}_0(T)} q^{2(\beta_1(s)-\mathrm{rot}_1(T,s))} W_1^\circ(s)$$

$$= (q + \bar{q})\, q^{-\omega(T)-\mathrm{rot}(T)} \sum_{s \in \mathrm{adm}_1(T)} q^{2(\beta_0(s)+\mathrm{rot}_0(T,s))} W_0^\circ(s).$$

Proof. It is not hard to see that $\mathrm{rot}_0(T, s) + \mathrm{rot}_1(T, s)$ is independent of the state s. It
is equal to the sum of rotation numbers of circles obtained by smoothing all crossings
of T, i.e. the rotation number $\mathrm{rot}(T)$ of T by definition. $\square$

To see how the Jones polynomial is related to the Alexander polynomial, let us
describe an expansion of the inverse of the Alexander polynomial. Consider all cycles
in our model of random walk which avoid the first arc A_1 on the knot diagram. Let
$\mathcal{Q}$ be the set of all such cycles which are primitive, i.e. they are not powers of any
other cycles. Recall that $\det(I - \mathcal{B})$ is, up to a factor of a power of t, the Alexander
polynomial of the knot in question. Given a cycle c, we will use $W(c)$ to denote its
weight. Then

$$(\det(I - \mathcal{B}))^{-1} = \prod_{c \in \mathcal{Q}} (1 - W(c))^{-1} = 1 + \sum_{k=1}^{\infty} \sum_{(c_1,\ldots,c_k) \in \mathcal{Q}^k} W(c_1) \cdots W(c_k).$$

This is the Foata-Zeilberger formula we mentioned in the introduction. For the con-
venience of readers, an exposition of this formula will be given in Section 4.

A k-tuple of cycles in $\mathcal{Q}^k$ is called *simple* if no edges are shared by cycles in this
k-tuple. Let $\mathcal{Q}^t$ be the set of all simple k-tuples of cycles, for $k = 1, 2, \ldots$. Given
$c \in \mathcal{Q}^t$, let $\beta_1(c)$ be the number of crossings in c, and $\mathrm{rot}(c)$ be the rotation number of
c. Note they are the same as the β_1 and rot_1 of the corresponding state in $\mathrm{adm}_0(T)$.

Finally, in order to have a one-one correspondence between $\mathrm{adm}_0(T)$ and $\mathcal{Q}^t$, we have to modify T slightly. The simplest way is to add a negative kink with rotation number -1 to the bottom of T and a positive kink with rotation number 1 to the top of T.

We will denote by $\mathcal{Q}^t_*$ the set of all simple k-tuples of cycles in the jump-down model.

Theorem 2.6. *With the 1-string link T appropriately chosen as described above, we have*

$$\frac{J(K)}{q + \bar{q}} = t^{\frac{\omega(T) - \mathrm{rot}(T)}{2}}(1 + \sum_{c \in \mathcal{Q}^t} t^{\mathrm{rot}(c) - \beta_1(c)} W(c))$$

$$= t^{\frac{\omega(T) + \mathrm{rot}(T)}{2}}(1 + \sum_{c \in \mathcal{Q}^t_*} \bar{t}^{(\mathrm{rot}(c) + \beta_0(c))} W(c)).$$

Comparing with the expansion of the Alexander polynomial, we see that the Jones polynomial $J(K)$ uses the summands $W(c_1) \cdots W(c_k)$ where no edges are repeated in the collection of cycles $c_1, \ldots, c_k$. A simple idea is that collections of cycles with repeated edges in the expansion of the Alexander polynomial might be lifted to collection of simple cycles on the cabling of T. This idea is realized in Theorem 3.3. In the next section, we will first generalize our discussion about the Jones polynomial to the colored Jones polynomial.

3. Limit of the colored Jones polynomial

3.1. **State sum for the colored Jones polynomial.** The set of finite dimensional irreducible representations of $\mathfrak{sl}(2)$ (or rather, the quantum group $U_q\mathfrak{sl}(2)$) can be listed as $V_1, V_2, V_3, \ldots$, where V_d is d-dimensional. The fundamental representation is V_2, which is the one used to construct the Jones polynomial $J(K)$. Other representations can also be used to construct knot polynomials. The knot polynomial obtained by "coloring" the (zero framed) knot K with the irreducible representation V_d is called the colored Jones polynomial $J(K, V^d)$ [9]. We have $J(K, V_1) = 1$ and $J(K, V_2) = J(K)$. And if K is the unknot,

$$J(K, V_d) = [d] = \frac{q^d - \bar{q}^d}{q - \bar{q}}.$$

We may also color K by non-irreducible representations, for example, by $V_2^{\otimes d}$. Such a colored Jones polynomial can be interpreted in two ways:

1. Assume that K has zero framing, let K^d be the link obtained by replacing K with d parallel copies (this is the zero framing cabling operation), then $J(K, V_2^{\otimes d}) = J(K^d)$.

2. We have the following relation in the representation ring of $\mathfrak{sl}(2)$: $V_2 \otimes V_d = V_{d+1} \oplus V_{d-1}$. Thus, $V_2^{\otimes d}$ is a linear combination of the irreducible modules V_{d+1}, V_{d-1}, V_{d-3}, ... and $J(K, V_2^{\otimes d})$ is the same linear combination of $J(K, V_{d+1})$, $J(K, V_{d-1})$, $J(K, V_{d-3})$, ...

These two interpretations can be used to establish a precise relation between the colored Jones polynomials and the cablings of the Jones polynomial. We quote from [5] such a relation in the case considered here:

$$J(K, V_{d+1}) = \sum_{j=0}^{d/2} (-1)^j \binom{d-j}{j} J(K^{d-2j}).$$

The decomposition $V_2 \otimes V_d = V_{d+1} \oplus V_{d-1}$ can be given explicitly in terms of the standard bases of these irreducible representations [6]. Suppose the standard basis of V_2 is $\{e_0, e_1\}$, and the standard basis of V_{d+1} is $\{f_0, f_1, \ldots, f_d\}$, then we have

$$f_0 = a \cdot e_0 \otimes e_0 \otimes \cdots \otimes e_0 \in V_2^{\otimes d},$$
$$f_d = b \cdot e_1 \otimes e_1 \otimes \cdots \otimes e_1 \in V_2^{\otimes d},$$

where a, b are products of q-analogue of Clebsch-Gordan coefficients [6].

For a 1-string link T, if it is colored by V_{d+1}, we get an invariant $F(T)$ which is a $U_q\mathfrak{sl}(2)$-morphism of V_{d+1}. Since V_{d+1} is an irreducible $U_q\mathfrak{sl}(2)$-module, we have

$$F(T)(f_i) = \lambda f_i, \qquad i = 0, 1, \ldots, d.$$

Furthermore, let K be the closure of T, then

$$J(K, V_{d+1}) = [d+1] \cdot \lambda.$$

On the other hand, if we color T by $V_2^{\otimes d}$, we may write the induced $U_q\mathfrak{sl}(2)$-morphism $F(T)$ on $V_2^{\otimes d}$ as follows:

$$F(T)(e_{i_1} \otimes \cdots \otimes e_{i_d}) = \sum_{j_1, \ldots, j_d} \int_{i_1 \cdots i_d}^{j_1 \cdots j_d} (T) \, e_{j_1} \otimes \cdots \otimes e_{j_d}.$$

Thus, the following lemma holds, which generalizes Lemma 2.4.

Lemma 3.1. *We have $\int_{0\cdots0}^{0\cdots0}(T) = \int_{1\cdots1}^{1\cdots1}(T) = \lambda$ and $J(K, V_{d+1}) = [d+1] \int_{0\cdots0}^{0\cdots0}(T) = [d+1] \int_{1\cdots1}^{1\cdots1}(T)$.*

We now can extend Theorem 2.6 to $J(K, V_{d+1})$. Notice that we assume the writhe $\omega(T) = 0$ and T^d is the zero-framing d-cabling of T. We denote by $\mathrm{adm}_0(T^d)$ the set of admissible states on T^d which assign 0 to all the top and bottom edges. The notation $\mathrm{adm}_1(T^d)$ has the obvious meaning.

Lemma 3.2. *With the notations as above, we have*

$$J(K, V_{d+1}) = [d+1] \, q^{\mathrm{rot}(T^d)} \sum_{s \in \mathrm{adm}_0(T^d)} q^{2(\beta_1(s) - \mathrm{rot}_1(T^d, s))} W_1^\circ(s)$$

$$= [d+1] \, \bar{q}^{\mathrm{rot}(T^d)} \sum_{s \in \mathrm{adm}_1(T^d)} q^{2(\beta_0(s) + \mathrm{rot}_0(T^d, s))} W_0^\circ(s).$$

Proof. Applying Turaev's state model to the tangle T^d, we get

$$\int_{i_1 \cdots i_d}^{j_1 \cdots j_d} (T) = (-q^2)^{w(T^d)} \sum_{s \in \mathrm{adm}_*(T^d)} q^{\mathrm{rot}_0(T^d, s) - \mathrm{rot}_1(T^d, s)} \Pi(s)$$

where $\mathrm{adm}_*(T^d)$ is the set of admissible states on T^d such that the bottom edges are assigned with $i_1, \ldots, i_d$ and top edges with $j_1, \ldots, j_d$, respectively. Then we can translate this expression for $\int_{i_1 \cdots i_d}^{j_1 \cdots j_d} (T)$ into the form appeared in Proposition 3.2 as we did in Section 2.4. □

Now the corresponding formula for $J(K, V_{d+1})$ of Theorem 2.6 is obtained from Theorem 2.6 by replacing $q + \bar{q} = [2]$ by $[d+1]$.

3.2. **Computation of the limit.** In this section, we prove our main theorem which calculates the limit of the renormalized colored Jones polynomials when the color parameter tends to infinity and the weight parameter tends to 1.

Theorem 3.3. *Let T be a 0-framed 1-string link, modified appropriately as in the Theorem 2.6, and K be the closure of T. Denote by $\mathcal{Q}$ ($\mathcal{Q}_*$) the set of primitive cycles in the jump-up (jump-down) model of random walk on T with $t = e^{-2h}$. Then*

$$\lim_{d \to \infty} \frac{J(K, V_{d+1})(e^{\frac{h}{d}})}{[d+1]} = \bar{t}^{\frac{\mathrm{rot}(T)}{2}} \left(1 + \sum_{(c_1, c_2, \ldots, c_k) \in \mathcal{Q}^k} W(c_1) W(c_2) \cdots W(c_k) \right)$$

$$= t^{\frac{\mathrm{rot}(T)}{2}} \left(1 + \sum_{(c_1, c_2, \ldots, c_k) \in \mathcal{Q}_*^k} W(c_1) W(c_2) \cdots W(c_k) \right).$$

Proof. Using the expansion of the colored Jones polynomials, it suffices to show that the weight of cycles $(c_1, c_2, \cdots, c_k)$ on T in the right-handed side is the limit of some cycles on T^d for large d. Let us compare the two jump-up models of random walks on T and T^d with $t = e^{-2h}$, and with $t = e^{-\frac{2h}{d}}$, respectively.

Consider first a simple cycle c on T. Recall that this is a cycle on T with no edges repeated. There are many ways to lift c to become a simple cycle $\tilde{c}$ on T^d. The reason for this multiplicity is that for each jump-up on c, we can choose one of the d over-crossed segments to jump up on T^d. In fact, if there are m jump-ups on c, there

will be m^d lifts $\tilde{c}$ on T^d. We need to calculate $\sum W(\tilde{c})$, a sum over all liftings of c. For a jump-up at a positive crossing on c, we get a (multiplicative) contribution $1 - e^{-2h}$ to $W(c)$. The corresponding contribution to $\sum W(\tilde{c})$ is a multiplicative factor

$$(1 - e^{-\frac{2h}{d}})(1 + e^{-\frac{2h}{d}} + e^{-\frac{4h}{d}} + \cdots + e^{-\frac{2(d-1)h}{d}}) = 1 - e^{-2h}.$$

Also, passing through an under-crossing on c contributes e^{-2h} to $W(c)$ and the corresponding contribution of $\tilde{c}$ is

$$(e^{-\frac{2h}{d}})^d = e^{-2h}.$$

Thus we have

$$\sum W(\tilde{c}) = W(c).$$

Obviously, $\beta_1(\tilde{c})$ and $\mathrm{rot}_1(T^d, \tilde{c})$ depend only on c. We also notice that $\mathrm{rot}(T^d) = d\,\mathrm{rot}(T)$. Thus,

$$\lim_{d \to \infty} (e^{\frac{2h}{d}})^{\frac{\mathrm{rot}(T^d)}{2}} \sum_{\tilde{c}} (e^{-\frac{2h}{d}})^{\mathrm{rot}_1(T^d, \tilde{c}) - \beta_1(\tilde{c})} W(\tilde{c})$$

$$= (e^{2h})^{\frac{\mathrm{rot}(T)}{2}} W(c).$$

Notice that the same argument holds true for a simple collection of cycles on T.

In general, given a non-simple collection of cycles c on T, we decorate each edge by an integer which is the number of times c traveling over that edge. There are only finitely many collections of cycles on T with a fixed decoration. For d sufficiently large, we can lift c to a simple collection of cycles on T^d. To get such a lifting, we will not have the freedom of jumping up onto any of the d over-crossed segments at a crossing. A particular jump-up at a crossing X on T has at most d liftings. For some other jump-up onto the segment going over X, we have to avoid the over-crossed segments jumped onto previously. There are at most d possible collisions for the liftings of these two jump-ups. Since

$$\lim_{d \to \infty} (1 - e^{\pm \frac{2h}{d}})(1 - e^{\pm \frac{2h}{d}})d = 0,$$

we conclude that in the limit when $d \to \infty$, the sum of weight of all non-simple liftings of c is zero. We may just do our calculation as if there are only simple liftings. Thus, the same calculation as we did before leads to

$$\lim_{d \to \infty} \sum_{\tilde{c}\ \mathrm{simple}} W(\tilde{c}) = W(c).$$

Finally, $\beta_1(\tilde{c})$ and $\mathrm{rot}_1(T^d, \tilde{c})$ are bounded by quantities depending only on c. Thus, we get

$$\lim_{d \to \infty} \frac{J(K, V_{d+1})(e^{\frac{h}{d}})}{[d+1]} = \bar{t}^{\frac{\mathrm{rot}(T)}{2}} \left(1 + \sum_{(c_1, c_2, \ldots, c_k) \in \mathcal{Q}^k} W(c_1) W(c_2) \cdots W(c_k) \right).$$

This finishes the proof of Theorem 3.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Ihara-Selberg zeta function and Melvin-Morton conjecture

4.1. **Lyndon words and the Foata-Zeilberger formula.** Let us recall the notion of Lyndon words and some results in [4]. For references to quoted results in this section, see [4].

Given a finite nonempty set X whose elements are totally ordered, we consider the monoid X^* generated by X. Let $<$ be the lexicographic order on X^* derived from the total order on X. A *Lyndon word* is defined to be a nonempty word in X^* which is prime, i.e. not the power of any other word, and is minimal in the class of its cyclic rearrangements. Let L denote the set of all Lyndon words. The following result is due to Lyndon.

Lemma 4.1. *Each nonempty word $w \in X^*$ can be uniquely written as a non-increasing juxtaposition of Lyndon words:*

$$w = l_1 l_2 \cdots l_m, \qquad l_i \in L, \ \ l_1 \geq l_2 \geq \cdots \geq l_m.$$

Let X be a finite set. Let $\mathcal{B}$ be a square matrix whose entries $b(x, x')$ $(x, x' \in X)$ form a set of commuting variables. For each Lyndon word $l \in L$, we associate with it a variable denoted by $[l]$. These variables $[l]$, $l \in L$, are assumed to be all distinct and commute with each other.

Given a word $w = x_1 x_2 \cdots x_k$ in X^*, define

$$\beta_{\mathrm{circ}}(w) = b(x_1, x_2) b(x_2, x_3) \cdots b(x_{k-1}, x_k) b(x_k, x_1)$$

and $\beta(w) = 1$ if w is the empty word. Notice that all the words in the same cyclic rearrangement class have the same β_{circ}-image. Also define

$$\beta([l]) = \beta_{\mathrm{circ}}(l)$$

for $l \in L$.

Now form the $\mathbb{Z}$-algebras of formal power series in the variables $[l]$ and $b(x, x')$ respectively. Extend β to a continuous homomorphism between these two $\mathbb{Z}$-algebras.

It makes sense to consider the product

$$\Lambda = \prod_{l \in L}(1 - [l])$$

as well as its inverse Λ^{-1}. We have

$$\beta(\Lambda) = \prod_{l \in L}(1 - \beta[l])$$

and

$$\beta(\Lambda^{-1}) = (\beta(\Lambda))^{-1}.$$

For a nonempty word $w \in X^*$, let it be written as in Lemma 4.1. Then define

$$\beta_{\mathrm{dec}}(w) = \beta_{\mathrm{circ}}(l_1)\beta_{\mathrm{circ}}(l_2)\cdots\beta_{\mathrm{circ}}(l_k).$$

If w is empty, $\beta_{\mathrm{dec}}(w) = 1$. Finally, define

$$\beta_{\mathrm{dec}}(X^*) = \sum_{w \in X^*} \beta_{\mathrm{dec}}(w).$$

The following theorem of Foata and Zeilberger is what we need.

Theorem 4.2. (Foata-Zeilberger formula) $\beta(\Lambda^{-1}) = \beta_{\mathrm{dec}}(X^*) = (\det(I - \mathcal{B}))^{-1}.$

This is a generalization of the Bowen-Lanford formula [3], which comes directly from the identity $\det(e^A) = e^{\mathrm{tr}A}$ for a matrix A.

4.2. **The Ihara-Selberg zeta function of a graph.** The Foata-Zeilberger formula in Theorem 3.2 is used in [4] to derive one of Bass' evaluations of the Ihara-Selberg zeta function for a graph [1]. For the reader's convenience, let us first recall Ihara's formulation of the zeta function in the original setting of Selberg (see [1]).

Let $\Gamma < PSL_2(\mathbb{R})$ be a uniform lattice (= discrete cocompact subgroup). An element $g \in \Gamma$ is hyperbolic if

$$l(g) = \min\{d(gx, x)\,;\, x \in \mathbb{R}_+^2\} > 0 \qquad (d = \text{Poincaré metric}).$$

Let $\mathcal{P}$ be the set of Γ-conjugacy classes of primitive hyperbolic elements in Γ, then the Ihara-Selberg zeta function is

$$Z(s) = \prod_{g \in \mathcal{P}}(1 - u^{l(g)})^{-1}, \qquad u = e^{-s}.$$

Let G be an directed graph with the set of edges $E(G) = \{e_1, e_2, \ldots, e_n\}$. Let S be an $n \times n$ matrix whose (i,j)-entry is equal to 1 if the terminal point of e_i is the same as the initial point of e_j, and 0 otherwise. On G, we may consider primitive cycles, which are oriented cycles formed by directed edges in the usual sense and which are

not powers of some other cycles. Let $\mathcal{C}$ be the set of primitive cycles on G. The Ihara-Selberg zeta function of G is

$$Z_G(u) = \prod_{c \in \mathcal{C}} (1 - u^{|c|})^{-1},$$

where $|c|$ is the length of the cycle c ($=$ the number of edges in c). The Foata-Zeilberger formula implies

$$Z_G(u) = (\det (I - uS))^{-1}.$$

If G is an undirected graph, in [1], Bass transformed G into an directed graph G' by giving each edge of G two different orientations and thinking of them as different directed edges. To study primitive, reduced cycles on G, where "reduced" means that an edge will not be traveled twice successively, Bass looked at the matrix $T = S - J$, where S is the matrix we defined in the previous paragraph for G' and J is the matrix whose (i, j)-entry is 1 if the i-th and j-th edges of G' come from the same edge of G, and 0 otherwise. Now let $\mathcal{R}$ be the set of primitive, reduced cycles on G, define

$$Z_G(u) = \prod_{c \in \mathcal{R}} (1 - u^{|c|})^{-1}.$$

One of Bass' evaluations of $Z_G(u)$, which is now a consequence of the Foata-Zeilberger formula, is

$$Z_G(u) = (\det (I - uT))^{-1}.$$

The Foata-Zeilberger formula is general enough so that we may apply it to Markov processes with a finite set of states. A cycle now will be a sequence of transitions of states from and back to a given one. In particular, in our model of random walk on a knot diagram discussed in Sections 2.1 and 2.2, we have the set of states $\{A_1, A_2, \ldots, A_n\}$ and the transition matrix $\tilde{\mathcal{B}}$. This case is degenerate since $\det (I - \tilde{\mathcal{B}}) = 0$. Nevertheless, we may consider all cycles in our model of random walk which avoid the first arc A_1 on the knot diagram. Let $\mathcal{Q}$ be the set of all such cycles which are primitive, then the Foata-Zeilberger formula implies

$$\prod_{c \in \mathcal{Q}} (1 - W(c))^{-1} = (\det (I - \mathcal{B}))^{-1},$$

where $W(c)$ is the weight of the cycle c and $\mathcal{B}$ is obtained from $\tilde{\mathcal{B}}$ by deleting the first row and column. Notice that $\det (I - \mathcal{B})$ is, up to a factor of a power of t, the Alexander polynomial of the knot in question. So we see that the inverse of the Alexander polynomial is an Ihara-Selberg type zeta function.

We have

$$\prod_{c \in \mathcal{Q}} (1 - W(c))^{-1} = 1 + \sum_{k=1}^{\infty} \sum_{(c_1, \ldots, c_k) \in \mathcal{Q}^k} W(c_1) \cdots W(c_k).$$

Hence we obtain the following expansion of the inverse of the Alexander polynomial:

Theorem 4.3.

$$(det\,(I - \mathcal{B}))^{-1} = \prod_{c \in \mathcal{Q}}(1 - W(c))^{-1} = 1 + \sum_{k=1}^{\infty} \sum_{(c_1,\ldots,c_k) \in \mathcal{Q}^k} W(c_1) \cdots W(c_k).$$

4.3. **Melvin-Morton function and Melvin-Morton Conjecture.** In [8], Melvin and Morton studied the dependence of the colored Jones polynomial on the "color" (that is the dimension d). They observed that

$$\frac{J(K, V_{d+1})(e^h)}{[d + 1]} = \sum_{m \geq 0,\, j \leq m} a_{jm}(K) d^j h^m.$$

Furthermore, Melvin and Morton conjectured that the function (which will be called the *Melvin-Morton function*)

$$M(K)(h) = \sum_{m \geq 0} a_{mm}(K) h^m$$

is the inverse of the Alexander polynomial.

Rozansky then was able to give a proof of this conjecture, on the level of rigor of physics, based essentially on calculating the limit

$$\lim_{d \to \infty} \frac{J(K, V_{d+1})(e^{\frac{h}{d}})}{[d + 1]}$$

and the known relationship between the semi-classical limit of Witten's Chern-Simons path integral and the Ray-Singer torsion. Rozansky's work went beyond the particular simple Lie algebra $\mathfrak{sl}(2)$ and extended the Melvin-Morton conjecture to its full generality.

The first rigorous proof of the Melvin-Morton conjecture was given by Bar-Natan and Garoufalidis [2]. Their proof used the full power of the theory of finite type knot invariants, together with some quite complicated combinatorial arguments. Later, Vaintrob and others simplified the combinatorial arguments of Bar-Natan and Garoufalidis (see, for example, [12]).

The Melvin-Morton conjecture can be deduced now as follows. By Theorem 3.3 and Theorem 4.3,

$$\lim_{d \to \infty} \frac{J(K, V_{d+1})(e^{\frac{h}{d}})}{[d + 1]} = \frac{t^{\frac{-\text{rot}(T)}{2}}}{\det(I - \mathcal{B})}.$$

On the other hand, it is easy to see that

$$\lim_{d \to \infty} \frac{J(K, V_{d+1})(e^{\frac{h}{d}})}{[d + 1]} = M(K)(h).$$

Hence the Melvin-Morton conjecture follows:

Theorem 4.4. *For any knot K which is the closure of a 0-framed 1-string link T,*

$$M(K)(h) = \frac{\bar{t}^{\frac{\text{rot}(T)}{2}}}{\det(I - \mathcal{B})},$$

where $t = e^{-2h}$.

Note the right side is the inverse of the symmetric Alexander polynomial of K when the 1-string link T is chosen appropriately as in Theorem 2.6.

Remark: In Theorem 3.3, we are actually calculating the limit of the partition function $\int_{0\cdots0}^{0\cdots0}(T)$ with a fixed boundary condition. This is rather like the calculation in statistical mechanics (e.g the limit of the Ising model). In statistic mechanics, the discontinuities of the limiting function are related with phase transitions. Thus, it might make sense to ask whether the zeros of the Alexander polynomial are of any significance and could be "observed".

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